THE HILBERT SCHEME OF POINTS IN THE PLANE IS CONNECTED

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This is a difficult topic to summarize briefly. I decided to assume a fair amount of background and leave numerous definitions as "black boxes" while still making literally true claims. The general discussion largely follows the intro of [Nak99] as well as parts of Lothar Göttsche's segment of [FGI$^+$05]. The final examples are discussed in more detail in [MS05]. Grothendieck's original construction is in [Gro62].

1. Hilbert Schemes

Definition 1.1. Let X be a projective scheme over an algebraically closed field k. The *Hilbert scheme* of X is the scheme Hilb_X characterized by the following property. The set of k-scheme morphisms $U \to \mathrm{Hilb}_X$ is in bijection with closed k-subschemes $Z \subset X \times U$ such that the induced projection map $Z \to U$ is flat. Moreover, this bijection is contravariantly functorial.

In fact, Hilb_X is projective. The case $U = \operatorname{Spec} k$ is instructive, since then Z is literally the set of closed k-subschemes of X. In this context the k-rational points and the closed points of Hilb_X coincide. This is a rigorous version of the slogan that the Hilbert scheme of X parameterizes the closed subschemes of X. For general U, we intuitively imagine Z to be the "graph" of a continuously deforming family of subschemes of X.

We can refine the construction in two distinct ways. First, for each Z above, we can associate a *Hilbert polynomial* $P(t) \in \mathbb{Q}[t]$ to each fiber $\pi^{-1}(u)$. We may then restrict to using only those Z above whose fibers have a fixed Hilbert polynomial $P(t)$, resulting in the closed subscheme Hilb_X^P. On the other hand, we may pick an open subscheme Y of X and restrict to $Z \subset Y \times U$ above, resulting in an open subscheme Hilb_Y^P of Hilb_X^P.

Definition 1.2. Let X be a quasiprojective k-scheme. Let $P(t) = n$ be constant. The *Hilbert scheme of n points in X* is

$$X^{[n]} := \mathrm{Hilb}_X^P .$$

The name arises from the fact that, given n closed points $x_1, \ldots, x_n \in X$, the closed subscheme $Z = \{x_1, \ldots, x_n\}$ is a closed point of $X^{[n]}$. Indeed, these points form an open subset of $X^{[n]}$. As we will see, the "interesting" part of the Hilbert scheme is the complement of this set.

Date: May 30, 2017.

2. A First Order Approximation

Definition 2.1. Let X be a quasiprojective k-variety. The nth *symmetric product* of X is the quasiprojective k-variety informally described as

$$X^{(n)} := X \times \cdots \times X/S_n$$

where S_n acts by permuting factors. If X is affine, this is the k-scheme $\mathrm{Spec}((k[X]^{\otimes n})^{S_n})$ where $k[X]$ is the coordinate ring of X.

It can be shown that $X^{(n)}$ is the *geometric quotient* of X by S_n, i.e. that it satisfies an appropriate universal property. By definition, the orbits of $X^{(n)}$ are orbits of n-tuples of elements of X. This gives rise to a stratification

$$X^{(n)} = \coprod_{\nu \vdash n} X^{(\nu)}$$

where $X^{(\nu)}$ consists of those multisubsets of n elements of X whose multiplicities are described by ν. When $\nu = (1^n)$ we recover n-element subsets of X, which is an open subset of $X^{(n)}$. Informally, $X^{(n)}$ is a "first order" approximation to a moduli space of n points in X. Geometric quotients like the above "rarely" exist as schemes while $X^{[n]}$ is a special case of a vastly more general construction.

Theorem 2.2. *Let X be a smooth quasiprojective variety over k. There is a surjective morphism of k-schemes*

$$\pi \colon X^{[n]}_{\mathrm{red}} \to X^{(n)}$$

called the Hilbert-Chow morphism, given on the level of points by

$$\pi(Z) := \sum_{x \in X} \mathrm{length}(Z_x)[x].$$

In fact, π is an isomorphism from $X^{((1^n))}$ to its inverse image. In this sense, the Hilbert scheme $X^{[n]}$ and the first order approximation $X^{(n)}$ agree "except at the edges."

3. The First Two Interesting Cases

When X is a smooth quasiprojective curve, we have $X^{[n]} = X^{(n)}$. For instance, the Hilbert scheme of points in the line $X = \mathbb{A} := \mathrm{Spec}\, k$ is informally

$$\begin{aligned} \mathbb{A}^{[n]} &= \{I \subset k[t] \mid I \text{ is an ideal}, \dim_k k[t]/I = n\} \\ &= \{f(z) \in k[z] \mid z^n + a_1 z^{n-1} + \cdots + a_n, a_i \in k\} \\ &= \mathbb{A}^{(n)}. \end{aligned}$$

This is perhaps the first non-trivial example of a Hilbert scheme: we've used the second-simplest possible variety (a line instead of a point) and we've used the simplest family of Hilbert polynomials.

The second non-trivial example is then $(\mathbb{A}^2)^{[n]}$, the Hilbert scheme of n points in the plane. Indeed, a theorem of Fogarty says that if X is a smooth quasiprojective surface, then $X^{[n]}$ is smooth and irreducible, and π is a resolution of singularities. In fact, $X^{(n)}$ in this latter case is not smooth for $n \geq 2$, so in this case the two constructions genuinely differ and $X^{[n]}$ is "better-behaved."

The geometry of $(\mathbb{A}^2)^{[n]}$ is extraordinarily rich and connects immediately to the combinatorics of Young diagrams. To see this, note that, at the level of closed points,

$$(\mathbb{A}^2)^{[n]} = \{I \subset k[x,y] \mid I \text{ is an ideal}, \dim_k k[x,y]/I = n\}.$$

Fixing, say, lexicographic ordering with $x > y$, we have

$$\dim k[x,y]/I = \dim k[x,y]/\operatorname{LT}(I).$$

It is easy to see that the exponents (a,b) of the monomials $x^a y^b$ not in $\operatorname{LT}(I)$ form a lower order ideal in $\mathbb{Z}^2_{\geq 0}$ under the component-wise partial order, consisting of n elements. This is precisely a Young diagram with n boxes. Moreover every such Young diagram arises in this way for some unique monomial ideal I_λ (generated by the "outer corners").

Thus to each point of $(\mathbb{A}^2)^{[n]}$ we may associate an ideal I_λ, and in fact we may "continuously vary" the ideals in an appropriate sense to travel from I to I_λ inside $(\mathbb{A}^2)^{[n]}$. Since $V(I_\lambda) = \{(0,0)\}$ for all λ, the subscheme associated to I_λ is highly non-reduced, i.e. it is very much not the coordinate ring of a classical affine variety. Intuitively, it tracks extra information about the collisions of points as they all go to zero. Since we may also continuously deform $I \in (\mathbb{A}^2)^{[n]}$ which vanish at n distinct points into each other, we have informally arrived at the fact that $(\mathbb{A}^2)^{[n]}$ is connected.

Interestingly, some of the nice properties of the Hilbert scheme in dimension 1 and 2 begin to fail already at $n = 3$. For instance, the Hilbert-Chow morphism need not be a resolution of singularities, and the Hilbert scheme may have "unexpectedly" large dimension. It would nonetheless be interesting to see more combinatorial connections between plane partitions and $(\mathbb{A}^3)^{[n]}$, since plane partitions index the relevant monomial ideals. Everything I found in this direction concerned Haiman's use of $(\mathbb{A}^2)^{[n]}$ for the $n!$ conjecture.

References

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